

Minimal rational curves on complete toric manifolds

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1 Introduction

For a complete uniruled smooth variety X , let $\text{RatCurves}^n(X)$ be the normalized space of rational curves on X . For an irreducible component \mathcal{K} of $\text{RatCurves}^n(X)$, let $\rho : \mathcal{U} \rightarrow \mathcal{K}$ and $\mu : \mathcal{U} \rightarrow X$ be the associated universal family morphisms. An irreducible component \mathcal{K} of $\text{RatCurves}^n(X)$ is called a *dominating component* if μ is dominant and a *minimal component* if furthermore for a general point $x \in X$, the variety $\mu^{-1}(x)$ is complete. Members of a minimal component are called *minimal rational curves*. The *degree* of \mathcal{K} is the degree of the intersection of $-K_X$ with any member in \mathcal{K} . For a fixed minimal component \mathcal{K} and a general point $x \in X$, we define the tangent map: $\tau_x : \mathcal{K}_x := \rho(\mu^{-1}(x)) \dashrightarrow \mathbb{P}(T_x X)$ by $\tau_x(\alpha) = \mathbb{P}(T_x C)$, where $C = \mu(\rho^{-1}(\alpha))$ is smooth at x . We denote by \mathcal{C}_x the closure of the image of τ_x in $\mathbb{P}(T_x X)$, which is called the *variety of minimal rational tangents* (VMRT for short) of \mathcal{K} at the point $x \in X$.

It turns out that the projective geometry of $\mathcal{C}_x \subset \mathbb{P}(T_x X)$ encodes a lot of the geometrical properties on X , which can be a useful tool in solving a number of problems on uniruled varieties (see the surveys [Hw1], [Mok]). Thus for a given X , it is worthwhile to determine \mathcal{C}_x . This has been worked out for many examples when X has Picard number 1 (loc. cit.). However, not many cases with large Picard number have been investigated. The main result of this paper, Corollary 5, gives a description of \mathcal{C}_x for a complete toric manifold. This implies that the minimal components of $\text{RatCurves}^n(X)$ correspond bijectively to some special primitive collections (Theorem 7), which can be easily read off from the fan defining X . As an application, we get examples of complete toric manifolds which do not have any minimal component (Example 1). Also, we will be able to classify toric Fano manifolds admitting a minimal component of degree $n = \dim X$ (Proposition 11). Finally, in Section 5, we propose a conjectural upper bound for $\rho_X(\dim \mathcal{C}_x + 1)$

when X is a toric Fano manifold, which is motivated by a conjecture of Mukai (see [Ca2]).

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2 Some general results

Here we collect a couple of general facts for which we do not have good references. The first one is more or less obvious.

Lemma 1. *Let X be a complete variety on which a connected algebraic group G acts with an open orbit $X_0 \subset X$. Suppose the stabilizer $\text{Stab}_G(x) \subset G$ of a point $x \in X_0$ is connected. Then for any dominating component \mathcal{K} of $\text{RatCurves}^n(X)$, the subvariety $\mathcal{K}_x \subset \mathcal{K}$ is irreducible.*

Proof. G acts on the universal family $\rho : \mathcal{U} \rightarrow X$ of \mathcal{K} . This action descends to the finite morphism $\rho' : \mathcal{U}' \rightarrow X$ obtained by the Stein factorization of ρ . Since the stabilizer $\text{Stab}_G(x)$ is connected, it fixes a point $y \in \rho'^{-1}(x)$, i.e., $\text{Stab}_G(y) = \text{Stab}_G(x)$. It follows that \mathcal{U}' contains an open subset isomorphic to X_0 , i.e., ρ' is birational. This means that \mathcal{K}_x is irreducible. \square

For the next one, we need some notation. Let G be a connected algebraic group. For an irreducible closed algebraic subvariety $S \subset G$ which contains the identity $e \in G$, let $[S]$ be the subgroup of G generated by elements in S and let $\langle S \rangle$ be the smallest closed algebraic subgroup of G containing S . Clearly, $[S] \subset \langle S \rangle$.

Proposition 2. *Let G be a connected commutative algebraic group over the complex numbers. If S is a closed irreducible algebraic subvariety of G contained in an (not necessarily closed) analytic subgroup H of G , then $\langle S \rangle = [S] \subset H$.*

Proof. We define \mathcal{W} to be the holomorphic foliation on G given by the cosets of H . Let \mathcal{S} be the family of subvarieties of G obtained by G -translates of S . Note that our assumption implies that every member of \mathcal{S} is contained in a leaf of \mathcal{W} .

Let $Z \subset G$ be a closed irreducible subvariety lying in a single leaf W of \mathcal{W} . Let $Z - S := \{z - s | z \in Z, s \in S\}$, which is an irreducible closed subvariety of G . We define $I(Z)$ to be the image of the group action: $(Z -$

$S) \times S \rightarrow G$, which is again an irreducible closed subvariety of G . Note that $I(Z)$ is the union of members of \mathcal{S} intersecting Z , which implies that $I(Z) \subset W$. Define inductively $I^{i+1}(Z) = I(I^i(Z))$. We have $I^i(Z) \subset I^{i+1}(Z) \subset W$. This implies that $I^n(Z) = I^m(Z)$ for $m \geq n = \dim G$.

Note that by the definition, if Z is contained in $[S]$, then $I(Z)$ is contained in $[S]$. Thus we have $I^m(S) \subset [S]$ for any $m \in \mathbb{N}$. On the other hand, each element of $[S]$ is contained in some $I^m(S)$, hence $[S] \subset I^n(S)$. Consequently, $[S] = I^n(S)$ is an algebraic variety, i.e., $[S]$ is an algebraic subgroup. It follows that $\langle S \rangle = [S] = I^n(S) \subset W$. \square

Corollary 3. *In Proposition 2, suppose there exists a Lie subalgebra \mathfrak{h} of the Lie algebra of G such that $S \subset \exp(\mathfrak{h})$, then $\dim \langle S \rangle \leq \dim \mathfrak{h}$.*

Note that Corollary 3 is a generalization of Lemma I.3.2 in [Zak] for abelian varieties. Our proof is completely different from Zak's.

3 Varieties of minimal rational tangents on a complete toric manifold

From now on, let X be a complete toric manifold of dimension n and $\mathbf{T} \subset X$ be the open orbit of the torus $(\mathbb{C}^*)^n$. A rational curve $C \subset X$ is called a *standard curve* if under the normalization $\nu : \mathbb{P}^1 \rightarrow C$, we have $\nu^*(TX) \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-p-1}$. It is easy to see that the deformations of C correspond to a dominating component \mathcal{K}^C of $\text{RatCurves}^n(X)$. As in Introduction, for $x \in \mathbf{T}$, we denote by \mathcal{K}_x^C the family of deformations of C passing through x . Then $\dim \mathcal{K}_x^C = p$ and \mathcal{K}_x^C is irreducible by Lemma 1. Just as before, we can define a variety of tangents $VT_x^C \subset \mathbb{P}(T_x X)$ by taking the closure of tangents at x of curves in \mathcal{K}_x^C which are smooth at x .

Theorem 4. *Let X be a complete toric manifold and $C \subset X$ a standard curve. Then the variety of tangents VT_x^C of \mathcal{K}^C at a $x \in \mathbf{T}$ is a linear subspace in $\mathbb{P}(T_x X)$.*

Proof. Let $D = X \setminus \mathbf{T}$ be the boundary divisor. It is well-known that $\Omega_X(\log D) \simeq \mathcal{O}_X^{\oplus n}$ (Prop. 3.1 [Oda]). Given $x \in \mathbf{T}$, we may identify \mathbf{T} with the torus such that x corresponds to the identity $e \in (\mathbb{C}^*)^n$. For an irreducible and reduced curve C passing through e , let $\langle C \rangle$ be the smallest toric subvariety of X containing C . Then

$$\langle C \rangle \cap \mathbf{T} = \langle C \cap \mathbf{T} \rangle$$

where the right hand side is in the sense of Proposition 2. Let V be the subspace of $H^0(X, \Omega_X(\log D)) \simeq \mathbb{C}^n$ consisting of vectors which annihilate the tangent vectors TC along C . The vector space V contains the space $H^0(C, T_X^*|_C) \subset H^0(C, \Omega_X(\log D)|_C) = H^0(X, \Omega_X(\log D))$. As C is a standard curve, the space $H^0(C, T_X^*|_C)$ has dimension $n - p - 1$, thus we get $\dim V \geq n - p - 1$. By Corollary 3, this implies $\dim \langle C \rangle \leq n - \dim V \leq p + 1$.

Let $\mathcal{K}^o \subset \mathcal{K}_e^C$ be the open subset consisting of standard curves. Denote by $\text{locus}(\mathcal{K}^o)$ the closure of the union of members of \mathcal{K}^o . Then $\text{locus}(\mathcal{K}^o)$ is a $(p+1)$ -dimensional constructible subset of X . Consider the p -dimensional family of toric subvarieties $\{\langle C_t \rangle\}_{t \in \mathcal{K}^o}$ which pass through the fixed point e . Since there is no positive-dimensional family of toric subvarieties fixing a point, we have $\langle C_t \rangle = \langle C_{t'} \rangle$ for two general points $t, t' \in \mathcal{K}^o$. Thus $\text{locus}(\mathcal{K}^o) \subset \langle C_t \rangle$ for some general $t \in \mathcal{K}^o$. Since $\dim(\text{locus}(\mathcal{K}^o)) = p + 1$, while $\dim(\langle C_t \rangle) \leq p + 1$, we have $\langle C_t \rangle = \text{locus}(\mathcal{K}^o)$ in a neighborhood of e . In particular, we have $VT_e^C = \mathbb{P}(T_e \langle C_t \rangle)$ which is linear since the toric subvariety $\langle C_t \rangle$ is smooth at e . \square

Remark 1. *The same argument works for singular complete toric varieties if one assumes that the standard curve C is contained in the smooth locus of X .*

Remark 2. *Let Y^n be a smooth projective variety dominated by a toric variety X , i.e. there exists a surjective morphism $\pi : X \rightarrow Y$. If $\rho_Y = 1$, then by Theorem 1 [OW], Y is isomorphic to \mathbb{P}^n . In Section 1 [OW], they raised the question whether Y is always toric. The answer is not always affirmative. In fact, let E be a vector bundle over a projective toric manifold X , take an ample line bundle L^{-1} such that $E \otimes L^{-1}$ is globally generated. This gives a surjective map $L^{\oplus d} \rightarrow E$ for some integer d . Then we obtain a surjective map $\mathbb{P}^{d-1} \times X \rightarrow \mathbb{P}(E)$, while $\mathbb{P}(E)$ is not necessarily a toric variety. We do not know any other example of projective manifolds dominated by a toric variety. Note that we may assume that π is generically finite by taking Stein factorization and resolution. Then, using a similar argument as in the proof of the precedent theorem and that of Prop. 2 [HM], we can show that for any standard curve C in Y , the variety of tangents VT_y^C at a general point $y \in Y$ is a disjoint union of linear subspaces. Thus Y contains projective spaces with trivial normal bundles. But these projective spaces do not always give rise to a projective bundle structure on Y .*

Corollary 5. *Let X be a complete toric manifold and let \mathcal{K} be a minimal component of degree $p + 2$. The variety of minimal rational tangents \mathcal{C}_x at a general point $x \in X$ is an irreducible linear subspace. The locus $\text{locus}(\mathcal{K}_e)$*

of curves in \mathcal{K} passing through the identity $e \in \mathbf{T}$ is a toric subvariety in X , which is isomorphic to \mathbb{P}^{p+1} with trivial normal bundle.

Proof. As is well-known, a general member of \mathcal{K} is a standard curve. Thus Theorem 4 implies that \mathcal{C}_x is an irreducible linear subspace for $x \in X$ general. As shown in [Ara] (Lemma 3.3), $\text{locus}(\mathcal{K}_e)$ is an immersed \mathbb{P}^{p+1} with trivial normal bundle. By the proof of Theorem 4, the subvariety $\text{locus}(\mathcal{K}_e)$ is equal to $\langle C \rangle$ for a general curve C in \mathcal{K}_e , thus it is a toric subvariety in X . Being a toric subvariety, the variety $\text{locus}(\mathcal{K}_x)$ is itself normal, thus it is smooth and isomorphic to \mathbb{P}^{p+1} . \square

Corollary 6. *Let X be a complete toric manifold and let \mathcal{K} be a minimal component of $\text{RatCurves}^n(X)$ of degree $p + 2$. Then there exist an open subset X_0 containing the torus \mathbf{T} , a smooth variety U and a \mathbb{P}^{p+1} -bundle structure $\phi_0 : X_0 \rightarrow U$ such that any curve in \mathcal{K} meeting X_0 is a line on a fiber of ϕ_0 .*

Proof. By Theorem 1.1 [Ara], there exists such an open subset X_0 , which contains the image of the torus action $(\mathbb{C}^*)^n \cdot \mathbb{P}^{p+1}$. In particular, X_0 contains \mathbf{T} . \square

This corollary implies that every projective toric manifold contains an open dense subset which is a projective bundle. This seems to be a new result. In general, one cannot hope that ϕ_0 can be extended to a morphism of X in codimension 1, as shown by the blow-up of \mathbb{P}^2 at two points.

4 Combinatorial description of minimal rational curves

We are now going to relate minimal components on X to combinatorial data of the fan corresponding to X . The basic results on toric varieties can be found in [Oda]. Recall that X is described by a finite fan Σ in the vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$, where N is a free abelian group of rank $n = \dim X$. As X is smooth and complete, the support of Σ is the whole space $N_{\mathbb{Q}}$ and every cone in Σ is generated by a part of a basis of N . For any i , we denote by $\Sigma(i)$ the set of all i -dimensional cones in Σ . For each $\sigma \in \Sigma(1)$, we take a primitive generator of $\sigma \cap N$. We denote by $G(\Sigma)$ the set of all such generators, which is in bijection with all 1-dimensional cones in Σ . The Picard number ρ_X of X is given by $\#G(\Sigma) - n$ (Cor. 2.5 [Oda]).

Definition 1 ([Ba1]). *i) A non-empty subset $\mathfrak{P} = \{x_1, \dots, x_k\}$ of $G(\Sigma)$ is called a primitive collection if for any i , the elements of $\mathfrak{P} \setminus \{x_i\}$ generate a $(k-1)$ -dimensional cone in Σ , while \mathfrak{P} does not generate a k -dimensional cone in Σ .*

ii) For a primitive collection $\mathfrak{P} = \{x_1, \dots, x_k\}$ of $G(\Sigma)$, let $\sigma(\mathfrak{P})$ be the unique cone in Σ which contains $x_1 + \dots + x_k$ in its interior. Let y_1, \dots, y_m be generators of $\sigma(\mathfrak{P})$, then there exists a unique equation with $a_i \in \mathbb{Z}_{>0}$:

$$x_1 + \dots + x_k = a_1 y_1 + \dots + a_m y_m.$$

This is the primitive relation associated to \mathfrak{P} . The degree of \mathfrak{P} is $\deg(\mathfrak{P}) = k - \sum_i a_i$. The order of \mathfrak{P} is k .

Theorem 7. *Let X be a complete toric manifold of dimension n . Then there is a bijection between minimal components of degree k on X and primitive collections $\mathfrak{P} = \{x_1, \dots, x_k\}$ of $G(\Sigma)$ such that $x_1 + \dots + x_k = 0$.*

Proof. If \mathcal{K} is a minimal component in $\text{RatCurves}^n(X)$ of degree k , by Corollary 5, there exists a smooth toric subvariety P in X which is isomorphic to \mathbb{P}^{k-1} . Let $\Sigma' \subset \Sigma$ be the sub-fan which gives P . Let $\mathfrak{P} := G(\Sigma') = \{x_1, \dots, x_k\}$. Then for any $x_i \in \mathfrak{P}$, the elements in $\mathfrak{P} \setminus \{x_i\}$ generate a $(k-1)$ -dimensional cone and we have $x_1 + \dots + x_k = 0$, which implies in particular that \mathfrak{P} does not generate a k -dimensional cone in Σ since every cone of Σ is generated by a part of a basis of N . This implies that \mathfrak{P} is a primitive collection of $G(\Sigma)$.

Now assume $\mathfrak{P} = \{x_1, \dots, x_k\}$ is a primitive collection such that $x_1 + \dots + x_k = 0$. Let Σ' be the sub-fan of Σ determined by \mathfrak{P} . Then Σ' defines a toric subvariety P of X which is isomorphic to \mathbb{P}^{k-1} . Let $D = \sum_{i=1}^d D_i$ be the decomposition of the boundary divisor $D = X \setminus \mathbf{T}$ into irreducible components, where $d = \rho_X + n$ is the number of elements in $G(\Sigma)$. We have a Euler-Jaczewski sequence (see for example Section 2.1 [OW]) as follows:

$$0 \rightarrow \Omega_X^1 \rightarrow \bigoplus_{i=1}^d \mathcal{O}_X(-D_i) \rightarrow \mathcal{O}_X^{\oplus \rho_X} \rightarrow 0.$$

As P is primitive, there is no other elements of $G(\Sigma)$ in the vector space $\mathbb{Q}x_1 + \dots + \mathbb{Q}x_k$. This shows that $\mathcal{O}_X(-D_i)|_P \simeq \mathcal{O}_P$ if D_i is not a divisor corresponding to one of the one dimensional cones in Σ generated by x_1, \dots, x_k . The precedent exact sequence gives

$$0 \rightarrow \Omega_X^1|_P \rightarrow \mathcal{O}_P(-1)^k \oplus \mathcal{O}_P^{\oplus \rho_X + n - k} \rightarrow \mathcal{O}_P^{\oplus \rho_X} \rightarrow 0.$$

Combining with the usual Euler sequence for $P \simeq \mathbb{P}^{k-1}$, this implies that the normal bundle of P in X is trivial. Take a line C in P , then its normal

bundle in X is isomorphic to $\mathcal{O}(1)^{k-2} \oplus \mathcal{O}^{n-k+1}$, i.e. C is a standard curve. Using the action of $(\mathbb{C}^*)^n$, we get an irreducible family of rational curves \mathcal{K} , which is a minimal component in $\text{RatCurves}^n(X)$. \square

When X is a projective toric manifold, there always exists a minimal component in $\text{RatCurves}^n(X)$. Theorem 7 has the following corollary, which has been firstly proved by Batyrev (Prop. 3.2 [Ba1]) in a completely different way.

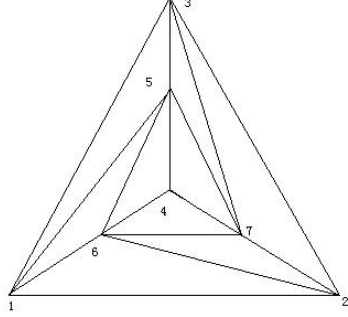
Corollary 8. *Let Σ be a fan which defines a projective toric manifold X . Then there exists a primitive collection $\mathfrak{P} = \{x_1, \dots, x_k\}$ such that $x_1 + \dots + x_k = 0$.*

Example 1. The assumption of projectivity of X in the precedent corollary is important as shown by the following example in Section 2.3 [Oda]. Let e_1, e_2, e_3 be a basis of \mathbb{Z}^3 . Let

$$e_4 = -e_1 - e_2 - e_3, \quad e_5 = -e_1 - e_2, \quad e_6 = -e_2 - e_3, \quad e_7 = -e_1 - e_3.$$

Let Σ be the complete regular fan in \mathbb{R}^3 obtained by joining 0 with the simplices of the triangulated tetrahedron in the displayed figure. We have $e_1 + e_2 + e_5 = 0$, however the set $\{e_1, e_2, e_5\}$ is not a primitive collection, since the cone generated by e_2, e_5 is not in Σ . Similarly we see that $\{e_2, e_3, e_6\}, \{e_1, e_3, e_7\}$ are not primitive collections. This implies that there is no minimal component in $\text{RatCurves}^n(X)$. This gives another way to see that the toric variety X defined by Σ is smooth complete but non-projective. The sub-fan in Σ generated by e_1, e_2, e_5 gives a toric subvariety which is isomorphic to $\mathbb{P}^2 \setminus \{pt\}$. If we denote by C the invariant curve corresponding to the cone generated by e_1, e_3 , then its cohomology class is given by $e_1 + e_2 + e_5 = 0$, which implies that its normal bundle in X is given by $\mathcal{O}(1) \oplus \mathcal{O}$, i.e. C is a standard curve. By Theorem 4, the variety of tangents VT_x^C determined by C is isomorphic to \mathbb{P}^1 , while all members of \mathcal{K}_x^C lie in an open set in $\mathbb{P}^2 \setminus \{pt\}$. If we denote by $\pi : Y \rightarrow X$ the blow-up of X along the invariant curve C , then Y is still non-projective since C deforms in X (Prop. 2 [Bon]). The fan $\Sigma(Y)$ of Y has a new element $e_0 = e_1 + e_3 = -e_7$. Thus the non-projective variety Y has a unique minimal component and its VMRT is just one point.

If one blows up X along the invariant curve corresponding to the cone generated by e_3 and e_7 , we obtain a projective variety X' (cf. Section 2.3 [Oda]). The fan Σ' of X' has a new element $e_8 = -e_1$ in $G(\Sigma')$. This implies that there exists a unique minimal component in $\text{RatCurves}^n(X')$ and its VMRT is just a point.



As an application of Theorem 7, we give an upper bound for the number of minimal components in $\text{RatCurves}^n(X)$.

Proposition 9. *Let X be a complete toric manifold of dimension n and Picard number ρ_X . For an integer p , we denote by n_p the number of minimal components in $\text{RatCurves}^n(X)$ of degree $p + 2$. Then we have*

i)

$$\sum_{p=0}^{n-1} n_p(p+2) \leq n + \rho_X.$$

ii) If n_p and n_q are non-zero for some integers p and q , then $p+q \leq n-2$.

iii) If $p \geq (n-1)/2$, then $n_p \leq 1$.

Proof. All three statements follow from the proof of Prop. 2.2 [Hw2], where it is shown that two linear subspaces in $\mathbb{P}(T_x X)$, which are components of VMRTs, have empty intersection in $\mathbb{P}(T_x X)$ for $x \in X$ general. For the claim i), a more combinatorial proof can be given as follows. Suppose we have two primitive collections $\mathfrak{P}_1 = \{x_1, \dots, x_{k+1}\}$ and $\mathfrak{P}_2 = \{y_1, \dots, y_{h+1}\}$ such that $x_1 + \dots + x_{k+1} = y_1 + \dots + y_{h+1} = 0$. If $\mathfrak{P}_1 \cap \mathfrak{P}_2$ is non-empty, we may assume $x_{k+1} = y_{h+1}$, then we get $x_1 + \dots + x_k = y_1 + \dots + y_h$. As x_1, \dots, x_k and y_1, \dots, y_h generate cones in Σ , this implies that the two cones are the same, thus $\mathfrak{P}_1 = \mathfrak{P}_2$. Now the claim i) follows from Theorem 7 and the fact that the number of elements in $G(\Sigma)$ is equal to $n + \rho_X$. \square

Remark 3. Even in dimension 2, one can construct many examples where the inequality in claim i) is an equality. If one restricts to toric Fano manifolds, the inequality in i) becomes an equality for products of copies of S_3 with projective spaces, where S_3 is the blow-up of \mathbb{P}^2 at three general points. In Example 4.7 [Sat], Sato constructed a toric Fano 4-folds with $\rho = 5$ by blowing up $\mathbb{P}^2 \times \mathbb{P}^2$, for which the inequality i) becomes an equality. It seems to be subtle to classify those such that i) becomes an equality.

Another application of Theorem 7 is the following. Recall that if X has a minimal component of degree $n+1$, then $X \simeq \mathbb{P}^n$ ([CMS]). The following proposition settles the next case when X is a toric Fano manifold. Recall that for a toric Fano manifold, every element in $G(\Sigma)$ is a primitive vector in N , and $G(\Sigma)$ is the set of vertices of a polytope Q and each facet of Q is the convex hull of a basis of N .

Lemma 10 (Lem. 3.3 [Ca1]). *Assume X is a toric Fano manifold. If Σ has two different primitive relations $x+y = z$ and $x+w = v$, then $w = -x-y$ and $v = -y$. Therefore there are at most two primitive collections of order 2 and degree 1 containing x , and the associated primitive relations are $x+y = (-w)$ and $x+w = (-y)$.*

Proposition 11. *Let X be a toric Fano manifold of dimension $n \geq 3$ which admits a minimal component of degree n . Then X is isomorphic to $\mathbb{P}^{n-1} \times \mathbb{P}^1$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ or a blow-up of a \mathbb{P}^{n-2} on $\mathbb{P}^{n-1} \times \mathbb{P}^1$. In particular, we have $\rho_X \leq 3$.*

Proof. Note that if $z_1, z_2 \in G(\Sigma)$ are two elements such that the two dimensional cone generated by them is not in Σ , then $\{z_1, z_2\}$ is a primitive collection. The Fano condition on X implies that either $z_1 + z_2 = 0$ or there exists another element $z \in G(\Sigma)$ such that $z_1 + z_2 = z$ (see for example [Ca1] page 1480).

By Theorem 7, the assumption implies that there is a primitive collection $\mathfrak{P} = \{x_1, \dots, x_n\}$ such that $x_1 + \dots + x_n = 0$. The vector space $H := \mathbb{R}x_1 + \dots + \mathbb{R}x_n$ divides $N \otimes_{\mathbb{Z}} \mathbb{R}$ into two sides. Assume that on one side, we have at least three elements y_1, y_2, y_3 in $G(\Sigma)$. Take an element $z \in G(\Sigma)$ on the other side of H , then $\{z, y_1\}, \{z, y_2\}, \{z, y_3\}$ are all primitive collections. By Lemma 10, up to reordering, we have $z + y_1 = 0$, $z + y_2 = (-y_3)$ and $-y_2, -y_3$ are all elements in $G(\Sigma)$. This gives $y_1 = y_2 + y_3$. We consider the primitive collections $\{-y_2, y_1\}, \{-y_2, y_3\}$. By Lemma 10, we obtain that $-y_2 + y_1 = -y_3$, which contradicts $y_1 = y_2 + y_3$. Thus there exist at most two elements on each side of H . Let y_1, y_2 be the two elements on one side of H and z_1, z_2 on the other side. If z_i is not in $\{-y_1, -y_2\}$, then we may apply Lemma 10, which shows that $-y_1, -y_2$ are in $G(\Sigma)$, a contradiction. Up to reordering, we may assume $z_1 = -y_1$ and $z_2 = -y_2$. Consider the primitive collections $\{-y_1, y_2\}$ and $\{-y_2, y_1\}$. Their primitive relations are $-y_1 + y_2 = x_i, -y_2 + y_1 = x_j$ for some i, j . This implies that $x_i + x_j = 0$, which is not possible since $n \geq 3$. In conclusion, the set $G(\Sigma)$ has at most $n+3$ elements, while $\rho_X = \#G(\Sigma) - n$. As a consequence, we have $\rho_X \leq 3$.

If $\rho_X = 3$, there are two elements y_1, y_2 on one side of H and an element z on the other side. Up to reordering, the precedent argument shows that

$z = -y_1$ and $-y_1 + y_2 = x_1$, i.e. $y_1 + x_1 = y_2$. This shows that X is the blow-up of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ along the invariant subvariety isomorphic to \mathbb{P}^{n-2} corresponding to the cone generated by x_1, y_1 .

If $\rho_X = 2$, i.e. $G(\Sigma)$ has $n + 2$ elements. On each side of H , there exists exactly one element in $G(\Sigma)$, say, y and z . As $\{y, z\}$ is a primitive collection, one has either $y + z = 0$ or $y + z = x_i$ for some i . The first case corresponds to $\mathbb{P}^{n-1} \times \mathbb{P}^1$ while the second fan corresponds to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. \square

5 A conjecture

For a toric Fano manifold X of dimension n , the pseudo-index ι_X is the smallest intersection number $-K_X \cdot C$ among all rational curves on X . In Theorem 1 [Ca2], it is proven that $\rho_X \leq 2n$ and $\rho_X(\iota_X - 1) \leq n$, which confirms a conjecture of Mukai. In analogous to this, we would like to propose the following

Conjecture 1. *For a toric Fano manifold X^n with $n \geq 3$, if there exists a minimal component \mathcal{K} of degree $p + 2$, then $\rho_X \cdot (p + 1) \leq n(n + 1)/2$.*

Note that the equality holds if $X \simeq (S_3)^d \times \mathbb{P}^{2d}$ or $(S_3)^d \times \mathbb{P}^{2d+1}$, where S_3 is the blow-up of \mathbb{P}^2 along three general points. In dimension 3, the equality also holds for the blow-up of $\mathbb{P}^2 \times \mathbb{P}^1$ along a \mathbb{P}^1 contained in \mathbb{P}^2 .

Since $\rho_X \leq 2n$ by [Ca2], we may assume that $p + 1 > \frac{n+1}{4}$ to check Conjecture 1. When $n = 3$, Conjecture 1 is immediate from Proposition 11. To check Conjecture 1 for $n = 4$, one only needs to show that if $p = 1$, then $\rho_X \leq 5$. We can use the classification of 4-dimensional toric Fano manifolds in [Ba2] to check that if $\rho_X \geq 6$, then $p = 0$, which shows that our conjecture holds for $n = 4$.

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